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A SIMPLIFIED FORMULA FOR IONOSPHERIC FARADAY ROTATION AT FREQUENCIES ABOVE 100 MC/S

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CONTENTS

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ABSTRACT

A simple formula for ionospheric Faraday rotation is derived starting from Maxwell's equations and using the following major assumptions: the ionosphere is a homogeneous gyrotropic medium with a scalar permeability and a tensor permittivity; the earth's magnetic field is constant and directed along the path of propagation; the ionosphere imposes no attenuation on the linearly polarized entering wave; and the carrier frequency is of the order of 100 Mc/s or higher.

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INTRODUCTION

When only an approximate estimation of ionospheric Faraday rotation is required for frequencies above 100 Mc/s, it is possible to derive a simplified formula for this purpose by making a number of assumptions. These assumptions lead to a model of the ionosphere which is idealized, and which is not an absolute representation of the spatial and temporal characteristics of the actual ionosphere. However, this model does permit rapid computation of an approximate value of Faraday rotation as a function of frequency, average ionospheric electron density, and average ionospheric magnetic flux density.

This paper presents the related assumptions, derivation, and use of this simplified formula.

ASSUMPTIONS

The ionosphere is assumed to be a gyrotropic medium. A gyrotropic medium is defined as an anisotropic medium in which the permittivity tensor is not symmetric. In the present case, the finite earth's magnetic field acting on the ionospheric charged particles brings about this asymmetry in the permittivity tensor.

The earth's magnetic field vector is assumed to be directed along the propagation path, and to have a constant magnitude along this path.

The ionospheric electron density is assumed to have a constant value along the propagation path.

The radiated wave entering the ionosphere is assumed to be a linearly polarized TEM (transverse electromagnetic) wave.

The carrier frequency is assumed to be in the order of 100 Mc/s or higher.

The permeability of the ionosphere is assumed to be a scalar quantity equal to the permeability of free space.

The ionosphere is assumed to exert no attenuation on the propagating wave.

SIMPLIFIED FORMULA

By use of the above assumptions, the following simplified formula for one-way ionospheric Faraday rotation may be derived as shown in the Appendix. Approximately, the per-unit-path-length Faraday rotation is given by:

$$\frac{\Omega}{z} = \frac{e\left(\frac{e}{m}\right)^2 BN}{2\,\epsilon_0\,c\omega^2}\,10^3 \text{ in radians/km} \tag{1}$$

where in the rationalized meter-kilogram-second (MKS) unit system:

e is electronic charge = 1.6×10^{-19} coulomb

m is electronic mass = 9.1×10^{-31} kilogram

ϵ_0 is permittivity of free space = 8.85×10^{-12} farad per meter

c is velocity of light = 3×10^8 meters per second

ω is carrier frequency in radians/sec

B is magnetic flux density in webers/m^2

N is electron density in electrons/m^3

Combining constants, the working formula in terms of f (cycles per sec) becomes:

$$\frac{\Omega}{z} = 1.355 \times 10^9 \left(\frac{BN}{f^2}\right) \text{ in degrees/km} \tag{1a}$$

RANGES OF VALUES FOR B AND N

The assumptions made in developing (1) involve a constant value for N and a constant value of B along the propagation path. In addition, the <u>direction</u> of the earth's magnetic field is assumed to be identical to the direction of propagation. These rather unrealistic assumptions require that the choices of the values of N and B be made with these assumptions in mind.

The value of B to be used in (1) should be an average one for that region of the earth's surface involved. World maps[1] of the earth's magnetic field intensity and direction can be used to pick a value of B for estimation of Faraday rotation. It is significant to note that the intensity of the magnetic field does not vary widely over the earth, and that an average value of 0.5×10^{-4} webers/m^2 is a good starting point in the absence of more detailed information, as shown in the footnote in the Appendix. Since it is actually the component of the earth's magnetic field colinear with the direction of propagation that gives rise[2] to the Faraday rotation given by (1), the above assumption of a completely colinear magnetic field will result generally in an upper-bound type of estimate of Faraday rotation.

The value of N to be used in (1) depends upon the thickness of the homogeneous spherical shell one wants to assume for the model of the ionosphere. It also depends upon the point in the sunspot cycle, and whether the local condition is day or night. As indicated in the footnote in the Appendix, for a shell that is assumed to range from 100 km to 600 km above the spherical earth, the daytime value of average electron density N varies[1] from 0.4×10^{12} to 10^{12} electrons/m^3 as the 11-year sunspot cycle goes from minimum to maximum. For a thinner shell model, correspondingly larger values of N must be used to be consistent with a constant value of the integral $\int N dh$, where h is height above the earth's surface.

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APPENDIX

DERIVATION OF THE SIMPLIFIED FORMULA

A linearly polarized TEM wave entering a gyrotropic medium will undergo a rotation of field components, and it is this rotation, as a function of the parameters of the medium and the frequency, that we wish to formulate.

A useful starting point[3] is the representation of the linearly polarized wave by the sum of two circularly polarized waves rotating in opposite directions. If a right-handed coordinate system of x, y, and z, with propagation in the positive z direction is used, then at $z = 0$, the electric field vector is

$$\overline{E} = \overline{a}_x E. \tag{2}$$

Let a counter-clockwise-rotating circularly polarized wave be given by:

$$\overline{E}_{ccw} = \overline{a}_x \frac{E}{2} + \overline{a}_y \, j \frac{E}{2}. \tag{3}$$

Let a corresponding clockwise-rotating circularly polarized wave be given by:

$$\overline{E}_{cw} = \overline{a}_x \frac{E}{2} - \overline{a}_y \, j \frac{E}{2}. \tag{4}$$

Then

$$\overline{E} = \overline{E}_{ccw} + \overline{E}_{cw}. \tag{5}$$

The Faraday rotation will be found from the phase changes in these two rotating waves as they propagate through the gyrotropic medium.

Applying Maxwells' equations to each rotating field separately, for the counter-clockwise wave

$$\nabla \times \overline{E}_{ccw} = -\mu_0 \frac{\partial \overline{H}_{ccw}}{\partial t} \tag{6}$$

Taking the curl of both sides,

$$\nabla \times (\nabla \times \overline{E}_{ccw}) = -\mu_0 \frac{\partial(\nabla \times \overline{H}_{ccw})}{\partial t} . \tag{7}$$

Now instead of using

$$\nabla \times \overline{H}_{ccw} = \overline{i} + \epsilon_0 \frac{\partial \overline{E}_{ccw}}{\partial t} , \tag{8}$$

use

$$\nabla \times \overline{H}_{ccw} = \hat{\epsilon} \frac{\partial \overline{E}_{ccw}}{\partial t} , \tag{9}$$

where $\hat{\epsilon}$ is a permittivity tensor which takes into account the interaction of the magnetic field of the medium with the charged particles therein.

But

$$\nabla \times (\nabla \times \overline{E}_{ccw}) = \nabla (\nabla \cdot \overline{E}_{ccw}) - \nabla^2 \overline{E}_{ccw} , \tag{10}$$

where

$$\nabla \cdot \overline{E}_{ccw} = \frac{\partial (E_{ccw})_x}{\partial x} + \frac{\partial (E_{ccw})_y}{\partial y} + \frac{\partial (E_{ccw})_z}{\partial z} . \tag{11}$$

Now since $\overline{E}_{ccw}$ is a field component of a TEM wave, there is no z component, and there is no spatial variation of $\overline{E}_{ccw}$ in the x or y direction. Thus $\nabla \cdot \overline{E}_{ccw} = 0$. Then

$$\nabla^2 \overline{E}_{ccw} = \mu_0 \hat{\epsilon} \frac{\partial^2 \overline{E}_{ccw}}{\partial t^2} . \tag{12}$$

Assume a sinusoidal time variation for E in (3);

$$E = E_0 e^{j\omega t} . \tag{13}$$

Substituting (13) and (3) into (12),

$$\nabla^2 \overline{E}_{ccw} = -\omega^2 \mu_0 \hat{\epsilon} \overline{E}_{ccw} . \tag{14}$$

Assuming the external magnetic field to be oriented in the positive z direction, the permittivity is given by:

$$\hat{\epsilon} = \epsilon_0 \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \tag{15}$$

with

$$\epsilon_{11} = \epsilon_{22} = 1 + \frac{\omega_p^2}{\omega_c^2 - \omega^2} \tag{16}$$

$$\epsilon_{12} = - \epsilon_{21} = \frac{j\,\omega_p^2 \left(\frac{\omega_c}{\omega}\right)}{\omega_c^2 - \omega^2} \tag{17}$$

$$\epsilon_{33} = 1 - \frac{\omega_p^2}{\omega^2} \tag{18}$$

where

ω_p is the plasma frequency $= \sqrt{(Ne/\epsilon_0)(e/m)}$

ω_c is the angular cyclotron frequency $= (e/m)\,B$.

When the counter-clockwise wave propagates through the gyrotropic medium, the expression for each field vector takes on a factor $e^{-\gamma_{ccw} z}$. Since there is no spatial variation in the x and y directions,

$$\nabla^2 \overline{E}_{ccw} = \frac{\partial^2}{\partial z^2} (\overline{E}_{ccw}). \tag{19}$$

Then (14) may be written as three equations:

$$\frac{\partial^2}{\partial z^2} \begin{bmatrix} E/2 \\ jE/2 \\ 0 \end{bmatrix} e^{-\gamma_{ccw} z} = - \omega^2 \mu_0 \epsilon_0 \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E/2 \\ j\,E/2 \\ 0 \end{bmatrix} e^{-\gamma_{ccw} z} \tag{20}$$

Differentiating either the first or second equation in (20) and dividing through by $E/2\ e^{-\gamma_{ccw} z}$,

$$(\gamma_{ccw})^2 = -\omega^2 \mu_0 \epsilon_0 (\epsilon_{11} + j\ \epsilon_{12}). \tag{21}$$

Assuming no attenuation in the wave, then $\gamma = j\beta$, and

$$\beta_{ccw} = \omega \sqrt{\mu_0 \epsilon_0} (\epsilon_{11} + j\ \epsilon_{12})^{1/2}. \tag{22}$$

By parallel procedure for the clockwise wave:

$$\beta_{cw} = \omega \sqrt{\mu_0 \epsilon_0} (\epsilon_{11} - j\ \epsilon_{12})^{1/2}. \tag{23}$$

Now it remains to express the Faraday rotation angle in terms of β_{ccw} and β_{cw}.

From (5) for any value z that places the wave within the gyrotropic medium, the electric field vector can be written:

$$\overline{E} = \overline{E}_{ccw}\ e^{-j\beta_{ccw} Z} + \overline{E}_{cw}\ e^{-j\beta_{cw} Z} \tag{24}$$

$$\overline{E} = \overline{a}_x E_x + \overline{a}_y E_y. \tag{25}$$

From (3) and (4)

$$E_x = \frac{E}{2} (e^{-j\beta_{ccw} Z} + e^{-j\beta_{cw} Z}). \tag{26}$$

$$E_y = \frac{E}{2} (j\ e^{-j\beta_{ccw} Z} - j\ e^{-j\beta_{cw} Z}). \tag{27}$$

Now the Faraday rotation angle is given by

$$\Omega = \tan^{-1} \frac{E_y}{E_x}. \tag{28}$$

Factoring out $e^{-j/2\,(\beta_{ccw} + \beta_{cw})z}$ from (26) and (27) obtains

$$E_x = E e^{-j/2\,(\beta_{ccw} + \beta_{cw})\,z} \cos \left(\frac{\beta_{cw} - \beta_{ccw}}{2}\right) z \tag{29}$$

$$E_y = -E e^{-j/2\,(\beta_{ccw} + \beta_{cw})\,z} \sin \left(\frac{\beta_{cw} - \beta_{ccw}}{2}\right) z \tag{30}$$

then

$$\frac{E_y}{E_x} = \frac{-\sin\left(\frac{\beta_{cw} - \beta_{ccw}}{2}\right) z}{\cos\left(\frac{\beta_{cw} - \beta_{ccw}}{2}\right) z}$$

or

$$\frac{E_y}{E_x} = -\tan\left(\frac{\beta_{cw} - \beta_{ccw}}{2}\right) z \tag{31}$$

then

$$\Omega = \left(\frac{\beta_{ccw} - \beta_{cw}}{2}\right) z. \tag{32}$$

Substituting (22) and (23) in (32):

$$\frac{\Omega}{z} = \frac{\omega}{2} \sqrt{\mu_0 \epsilon_0}\ \left[(\epsilon_{11} + j\ \epsilon_{12})^{1/2} - (\epsilon_{11} - j\ \epsilon_{12})^{1/2}\right]$$

or

$$\frac{\Omega}{z} = \frac{\omega}{2} \sqrt{\mu_0 \, \epsilon_0 \, \epsilon_{11}} \quad \left[\left(1 + j \, \frac{\epsilon_{12}}{\epsilon_{11}}\right)^{1/2} - \left(1 - j \, \frac{\epsilon_{12}}{\epsilon_{11}}\right)^{1/2} \right] . \tag{33}$$

From (16) and (17)

$$j \, \frac{\epsilon_{12}}{\epsilon_{11}} = - \frac{\omega_p^2 \left(\frac{\omega_c}{\omega}\right)}{\omega_c^2 - \omega^2 + \omega_p^2}$$

or

$$j \, \frac{\epsilon_{12}}{\epsilon_{11}} = \frac{\left(\frac{\omega_c}{\omega}\right)}{\left(\frac{\omega}{\omega_p}\right)^2 - \left(\frac{\omega_c}{\omega_p}\right)^2 - 1} . \tag{34}$$

If (34) is computed for a frequency of 100 Mc/s and some "typical" values* of $N = 10^{12}$ and $B = 0.5 \times 10^{-4}$, then $\omega_c = 8.8 \times 10^6$; $\omega_p = 5.66 \times 10^7$; $\omega = 6.28 \times 10^8$ and $j \, (\epsilon_{12}/\epsilon_{11}) = 1.15 \times 10^{-4}$.

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*From Johnson [1], the value of B ranges from 0.25×10^{-4} to 0.70×10^{-4} on the total intensity plot on page 121. Also from Johnson [1], on page 28, the integral of the electron density height profiles for daytime can be approximated as follows:

For minimum of sunspot cycle: $\int N dh \approx 2 \times 10^{14} \frac{\text{electrons}}{m^3} - \text{km}$

For maximum of sunspot cycle: $\int N dh \approx 5 \times 10^{14} \frac{\text{electrons}}{m^3} - \text{km}$

Then for a 500-km thick ionospheric shell with uniform electron density from 100 to 600 km above the earth:

For minimum of sunspot cycle: $N \approx 0.4 \times 10^{12} \frac{\text{electrons}}{m^3}$

For maximum of sunspot cycle: $N \approx 10^{12} \frac{\text{electrons}}{m^3}$.

The binomial expressions in (33) may then be expanded in series form, retaining just the first two terms of each series to a good approximation, as follows:

$$\left(1 + j\,\frac{\epsilon_{12}}{\epsilon_{11}}\right)^{1/2} \approx 1 + \frac{1}{2}\,j\,\frac{\epsilon_{12}}{\epsilon_{11}} \tag{35}$$

$$\left(1 - j\,\frac{\epsilon_{12}}{\epsilon_{11}}\right)^{1/2} \approx 1 - \frac{1}{2}\,j\,\frac{\epsilon_{12}}{\epsilon_{11}}. \tag{36}$$

Then

$$\frac{\Omega}{z} \approx \frac{\omega\sqrt{\mu_0\,\epsilon_0\,\epsilon_{11}}}{2}\left(j\,\frac{\epsilon_{12}}{\epsilon_{11}}\right). \tag{37}$$

Substituting for ϵ_{11} and ϵ_{12},

$$\frac{\Omega}{z} \approx -\,\frac{\sqrt{\mu_0\,\epsilon_0}\;\omega_p^2\,\omega_c}{2\,\left[(\omega_c^2 - \omega^2 + \omega_p^2)\,(\omega_c^2 - \omega^2)\right]^{1/2}}. \tag{38}$$

Note that the above approximation improves as the frequency increases. Now at the "typical" values for N and B discussed above, compare ω_c^2, ω_p^2 and ω^2 for a frequency of 100 Mc/s.

$$\omega_c^2 \;\square\; 7.74 \times 10^{13};\quad \omega_p^2 = 3.20 \times 10^{15};\quad \omega^2 = 3.94 \times 10^{17}$$

Therefore for these values:

$$\omega_c^2 << \omega_p^2 << \omega^2.$$

Then (38) becomes

$$\frac{\Omega}{z} \approx \frac{\sqrt{\mu_0 \epsilon_0}\, \omega_p^2\, \omega_c}{2\omega^2} . \tag{39}$$

Substituting for ω_p and ω_c and letting $\sqrt{\mu_0 \epsilon_0} = 1/c$, the simplified formula becomes

$$\frac{\Omega}{z} = \frac{e\left(\frac{e}{m}\right)^2 BN}{2\,\epsilon_0\, c\, \omega^2}\, 10^3 \text{ radians/km.} \tag{40}$$

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